Review of Testing Hypotheses

Basic Concepts of Testing Hypotheses

Null hypothesis (H_0) and alternative hypothesis (H_a)

The alternative hypothesis of a statistical test is the hypothesis that the researcher wishes to support.

The null hypothesis is the negation of the alternative hypothesis.

Test statistic

The test statistic of a statistical test is the sample statistic used to determine whether or not to reject H_0 .

Rejection region

The rejection region of a statistical test is the set of test statistic values that will lead to H_0 being rejected.

Decision and Conclusion

Decision to (or not to) reject H_0 based on the comparison of test statistic to rejection region.

Type I Error Rejecting H_0 when H_0 is true.

Type II Error Failing to reject H_0 when H_0 is false.

	True State of Nature	
	H_0 True	H_0 False
Reject H_0	Type I error	Correct decision
Fail to reject H_0	Correct decision	Type II error

α: Probability of committing a Type I error $\alpha = P(\text{rejecting } H_0 \mid H_0 \text{ true})$

β: Probability of committing a Type II error $P(\text{failing to reject } H_0 \mid H_0 \text{ false})$

Power $= 1 - \beta$

Level of significance

The upper bound of the probability of committing Type I error.

Commonly used level of significance: 0.1, 0.05, 0.025, 0.01, 0.005, $\cdots$

The p- value of a Statistical Test

p **- value**

The p-value of a statistical test is the probability that the test statistic reaches the observed value or gets more extreme value.

$H_0 : \mu = \mu_0 \quad H_a : \mu > \mu_0$

$p\text{-value} = P\left(Z \geq z_{(\text{obs})}\right) \quad \left(\text{or } p\text{-value} = P\left(t \geq t_{(\text{obs})}\right)\right)$

$H_0 : \mu = \mu_0 \quad H_a : \mu < \mu_0$

$p\text{-value} = P\left(Z \leq z_{(\text{obs})}\right) \quad \left(\text{or } p\text{-value} = P\left(t \leq t_{(\text{obs})}\right)\right)$

$H_0 : \mu = \mu_0 \quad H_a : \mu \neq \mu_0$

$p\text{-value} = 2P\left(Z \geq \left| z_{(\text{obs})} \right|\right) \quad \left(\text{or } p\text{-value} = 2P\left(t \geq \left| t_{(\text{obs})} \right|\right)\right)$

Reject H_0 if p- value $< \alpha$.

Example A random sample of size 100 has been drawn from a population distribution. It is known that the sample means is 59 and the sample standard deviations is 10. Find the *p*-value for the test $H_0 : \mu = 60$ vs. $H_a : \mu < 60$.

Example A random sample of size 100 has been drawn from a population distribution. It is known that the sample means is 59 and the sample standard deviations is 10. Find the *p*-value for the test $H_0 : \mu = 60$ vs. $H_a : \mu \neq 60$.

Review of Statistical Inference: One-sample Problems

Statistical Inference About μ

Statistical Inference About μ: σ^2 Known

Point estimator of μ: $\hat{\mu} = \bar{X}$

$\bar{X}$ is an unbiased estimator of μ.

Confidence interval for μ when σ^2 is known:

$$\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{(Large sample or normality is required.)}$$

Upper confidence limit: $\mu \le \bar{X} + z_{\alpha}\frac{\sigma}{\sqrt{n}}$

Lower confidence limit: $\mu \ge \bar{X} - z_{\alpha}\frac{\sigma}{\sqrt{n}}$

Statistical test about μ: (Large sample or normality is required.)

$H_0 : \mu = \mu_0$

H_a: A. $\mu > \mu_0$

B. $\mu < \mu_0$

C. $\mu \ne \mu_0$

Test Statistic: $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } Z > z_{\alpha} \\ \text{B. } Z < -z_{\alpha} \\ \text{C. } Z < -z_{\alpha/2} \text{ or } Z > z_{\alpha/2} \end{cases}$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(Z \ge z_{(\text{obs})}\right) \\ \text{B. } P\left(Z \le z_{(\text{obs})}\right) \\ \text{C. } 2P\left(Z \ge \left|z_{(\text{obs})}\right|\right) \end{cases}$$

Statistical Inference About μ: σ^2 Unknown

Point estimator of μ: $\hat{\mu} = \bar{X}$

$\bar{X}$ is an unbiased estimator of μ.

Confidence interval for μ when σ^2 is unknown:

$$\bar{X} - t_{\alpha/2,\,n-1}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2,\,n-1}\frac{S}{\sqrt{n}} \quad \text{(Normality is required.)}$$

Upper confidence limit: $\mu \le \bar{X} + t_{\alpha,\,n-1}\frac{S}{\sqrt{n}}$

Lower confidence limit: $\mu \ge \bar{X} - t_{\alpha,\,n-1}\frac{S}{\sqrt{n}}$

Statistical test about μ: (Normality is required.)

$H_0 : \mu = \mu_0$

$H_a :$ A. $\mu > \mu_0$

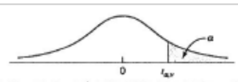
B. $\mu < \mu_0$

C. $\mu \neq \mu_0$

Test Statistic: $t = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } t > t_{\alpha,n-1} \\ \text{B. } t < -t_{\alpha,n-1} \\ \text{C. } t < -t_{\alpha/2,n-1} \text{ or } t > t_{\alpha/2,n-1} \end{cases}$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(t \geq t_{(\text{obs})}\right) \\ \text{B. } P\left(t \leq t_{(\text{obs})}\right) \\ \text{C. } 2P\left(t \geq \left|t_{(\text{obs})}\right|\right) \end{cases}$$



α

ν	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.727	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.49	4.019	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.20	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.992
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.255	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.254	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.254	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Statistical Inference About p

Point estimator of p: $\hat{p}$

$\hat{p}$ is an unbiased estimator of p.

Confidence interval for p:

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad \text{(Large sample is required.)}$$

Upper confidence limit: $p \le \hat{p} + z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Lower confidence limit: $p \ge \hat{p} - z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Statistical test about p: (Large sample is required.)

$H_0: p = p_0$

H_a: A. $p > p_0$

B. $p < p_0$

C. $p \neq p_0$

Test Statistic: $Z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } Z > z_\alpha \\ \text{B. } Z < -z_\alpha \\ \text{C. } Z < -z_{\alpha/2} \text{ or } Z > z_{\alpha/2} \end{cases}$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(Z \geq z_{(\text{obs})}\right) \\ \text{B. } P\left(Z \leq z_{(\text{obs})}\right) \\ \text{C. } 2P\left(Z \geq \left|z_{(\text{obs})}\right|\right) \end{cases}$$

Statistical Inference About σ^2

Point estimator of σ^2 : $\hat{\sigma}^2 = S^2$

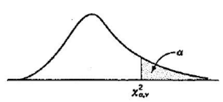
S^2 is an unbiased estimator of σ^2.

Confidence interval for σ^2:

$$\frac{(n-1)\,S^2}{\chi^2_{\alpha/2,\,n-1}} \le \sigma^2 \le \frac{(n-1)\,S^2}{\chi^2_{1-\alpha/2,\,n-1}} \quad \text{(Normality is required.)}$$

Upper confidence limit: $\sigma^2 \le \dfrac{(n-1)\,S^2}{\chi^2_{1-\alpha,\,n-1}}$

Lower confidence limit: $\sigma^2 \ge \dfrac{(n-1)\,S^2}{\chi^2_{\alpha,\,n-1}}$



	α								
v	0.995	0.990	0.975	0.950	0.500	0.050	0.025	0.010	0.005
1	0.00 +	0.00 +	0.00 +	0.00 +	0.45	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	1.39	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	2.37	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	3.36	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	4.35	11.07	12.38	15.09	16.75
6	0.68	0.87	1.24	1.64	5.35	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	6.35	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	7.34	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	8.34	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	9.34	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	10.34	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	11.34	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	12.34	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	13.34	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	14.34	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	15.34	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	16.34	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	17.34	28.87	31.53	34.81	37.16
19	6.884	7.63	8.91	10.12	18.34	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	19.34	31.41	34.17	37.57	40.00
25	10.52	11.52	13.12	14.61	24.34	37.65	40.65	44.31	46.93
30	13.79	14.95	16.79	18.49	29.34	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	39.34	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	49.33	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	59.33	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	69.33	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	79.33	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	89.33	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	99.33	124.34	129.56	135.81	140.17

Statistical test about σ^2: (Normality is required.)

$H_0 : \sigma^2 = \sigma_0^2$

H_a : A. $\sigma^2 > \sigma_0^2$

B. $\sigma^2 < \sigma_0^2$

C. $\sigma^2 \neq \sigma_0^2$

Test Statistic: $\chi^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } \chi^2 > \chi^2_{\alpha,n-1} \\ \text{B. } \chi^2 < \chi^2_{1-\alpha,n-1} \\ \text{C. } \chi^2 < \chi^2_{1-\alpha/2,n-1} \text{ or } \chi^2 > \chi^2_{\alpha/2,n-1} \end{cases}$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(\chi^2 \geq \chi^2_{(\text{obs})}\right) \\ \text{B. } P\left(\chi^2 \leq \chi^2_{(\text{obs})}\right) \\ \text{C. } 2\min\left\{P\left(\chi^2 \leq \chi^2_{(\text{obs})}\right), P\left(\chi^2 \geq \chi^2_{(\text{obs})}\right)\right\} \end{cases}$$

Review of Statistical Inference: Two-sample Problems

Statistical Inference About $\mu_1 - \mu_2$

Statistical Inference About $\mu_1 - \mu_2$: σ_1^2 and σ_2^2 Known

Point estimator of $\mu_1 - \mu_2$: $\bar{X}_1 - \bar{X}_2$

$\bar{X}_1 - \bar{X}_2$ is an unbiased estimator of $\mu_1 - \mu_2$.

Confidence interval for $\mu_1 - \mu_2$ when σ_1^2 and σ_2^2 are known:

$$\bar{X}_1 - \bar{X}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(Large sample or normality is required.)

Statistical test about $\mu_1 - \mu_2$: (Large samples or normality is required.)

$H_0 : \mu_1 - \mu_2 = \Delta_0$

H_a : A. $\mu_1 - \mu_2 > \Delta_0$

B. $\mu_1 - \mu_2 < \Delta_0$

C. $\mu_1 - \mu_2 \ne \Delta_0$

Test Statistic: $$Z = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } Z > z_\alpha \\ \text{B. } Z < -z_\alpha \\ \text{C. } Z < -z_{\alpha/2} \text{ or } Z > z_{\alpha/2} \end{cases}$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(Z \ge z_{(\text{obs})}\right) \\ \text{B. } P\left(Z \le z_{(\text{obs})}\right) \\ \text{C. } 2P\left(Z \ge \left| z_{(\text{obs})} \right|\right) \end{cases}$$

Statistical Inference About $\mu_1 - \mu_2$: $\sigma_1^2 = \sigma_2^2$ Unknown

Point estimator of $\mu_1 - \mu_2$: $\bar{X}_1 - \bar{X}_2$

$\bar{X}_1 - \bar{X}_2$ is an unbiased estimator of $\mu_1 - \mu_2$.

Confidence interval for μ_1-μ_2 when σ_1^2 and σ_2^2 are unknown $\left(\sigma_1^2 = \sigma_2^2\right)$:

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2, n_1+n_2-2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 + t_{\alpha/2, n_1+n_2-2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$s_p = \sqrt{\frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}} \quad \text{(Normality is required.)}$$

Statistical test about $\mu_1 - \mu_2$: (Normality is required.)

$H_0: \mu_1 - \mu_2 = \Delta_0$

H_a: A. $\mu_1 - \mu_2 > \Delta_0$

B. $\mu_1 - \mu_2 < \Delta_0$

C. $\mu_1 - \mu_2 \neq \Delta_0$

Test Statistic: $t = \dfrac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ $\left(s_p = \sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}} \right)$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } t > t_{\alpha, n_1+n_2-2} \\ \text{B. } t < -t_{\alpha, n_1+n_2-2} \\ \text{C. } t < -t_{\alpha/2, n_1+n_2-2} \text{ or } t > t_{\alpha/2, n_1+n_2-2} \end{cases}$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(t \geq t_{(\text{obs})}\right) \\ \text{B. } P\left(t \leq t_{(\text{obs})}\right) \\ \text{C. } 2P\left(t \geq \left|t_{(\text{obs})}\right|\right) \end{cases}$$

Statistical Inference About $\mu_1 - \mu_2$: $\sigma_1^2 \neq \sigma_2^2$ Unknown

Point estimator of $\mu_1 - \mu_2$: $\bar{X}_1 - \bar{X}_2$

$\bar{X}_1 - \bar{X}_2$ is an unbiased estimator of $\mu_1 - \mu_2$.

Confidence interval for μ_1-μ_2 when σ_1^2 and σ_2^2 are unknown $\left(\sigma_1^2 \neq \sigma_2^2\right)$:

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 + t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

(Normality is required.)

$$\left(\nu = \frac{\left(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1-1}+\frac{\left(S_2^2/n_2\right)^2}{n_2-1}} - 2\right)$$

Statistical test about $\mu_1 - \mu_2$: (Normality is required.)

$H_0: \mu_1 - \mu_2 = \Delta_0$

H_a: A. $\mu_1 - \mu_2 > \Delta_0$

B. $\mu_1 - \mu_2 < \Delta_0$

C. $\mu_1 - \mu_2 \neq \Delta_0$

Test Statistic: $t = \dfrac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}}$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } t > t_{\alpha,\nu} \\ \text{B. } t < -t_{\alpha,\nu} \\ \text{C. } t < -t_{\alpha/2,\nu} \text{ or } t > t_{\alpha/2,\nu} \end{cases} \qquad \left(\nu = \frac{\left(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1-1}+\frac{\left(S_2^2/n_2\right)^2}{n_2-1}} - 2\right)$$

$$p\text{-value} = \begin{cases} \text{A. } P\left(t \ge t_{(\text{obs})}\right) \\ \text{B. } P\left(t \le t_{(\text{obs})}\right) \\ \text{C. } 2P\left(t \ge \left|t_{(\text{obs})}\right|\right) \end{cases}$$

Example An article in the *Journal Hazardous Waste and Hazardous Materials* (Vol. 6, 1989) reported the results of an analysis of the weight of calcium in standard cement and cement doped with lead. Reduced levels of calcium would indicate that the hydration mechanism in the cement is blocked and would allow water to attack various locations in the cement structure. Ten samples of standard cement had an average weight percent calcium of $\bar{x}_1 = 90.0$, with a sample standard deviation of s_1=5.0, and 15 samples of the lead-doped cement had an average weight percent calcium of $\bar{x}_2$=87.0, with a sample standard deviation of s_2=4.0. Find a 95% confidence interval for the difference in means, $\mu_1 - \mu_2$, for the two types of cement. It is assumed that weight percent calcium is normally distributed and that that both normal populations have the same standard deviation.

Example (Continue) An article in the *Journal Hazardous Waste and Hazardous Materials* (Vol. 6, 1989) reported the results of an analysis of the weight of calcium in standard cement and cement doped with lead. Reduced levels of calcium would indicate that the hydration mechanism in the cement is blocked and would allow water to attack various locations in the cement structure. Ten samples of standard cement had an average weight percent calcium of $\bar{x}_1 = 90.0$, with a sample standard deviation of s_1=5.0, and 15 samples of the lead-doped cement had an average weight percent calcium of $\bar{x}_2$=87.0, with a sample standard deviation of s_2=4.0. Find a 95% confidence interval for the differenc in means, $\mu_1 - \mu_2$, for the two types of cement. It is assumed that weight percent calcium is normally distributed.

Statistical Inference About $p_1 - p_2$

Point estimator of $p_1 - p_2$: $\hat{p}_1 - \hat{p}_2$

$\hat{p}_1 - \hat{p}_2$ is an unbiased estimator of $p_1 - p_2$.

Confidence interval for $p_1 - p_2$:

$$\hat{p}_1 - \hat{p}_2 - z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$\le \; p_1 - p_2 \; \le \hat{p}_1 - \hat{p}_2 - z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

(Large samples are required.)

Statistical test about $p_1 - p_2$: (Large samples are required.)

$H_0: p_1 - p_2 = \Delta_0$

H_a: A. $p_1 - p_2 > \Delta_0$

B. $p_1 - p_2 < \Delta_0$

C. $p_1 - p_2 \neq \Delta_0$

Test Statistic: $Z = \dfrac{\hat{p}_1 - \hat{p}_2 - \Delta_0}{\sqrt{\dfrac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$

H_0 is rejected if $\begin{cases} \text{A. } Z > z_\alpha \\ \text{B. } Z < -z_\alpha \\ \text{C. } Z < -z_{\alpha/2} \text{ or } Z > z_{\alpha/2} \end{cases}$

$p\text{-value} = \begin{cases} \text{A. } P\left(Z \geq z_{(\text{obs})}\right) \\ \text{B. } P\left(Z \leq z_{(\text{obs})}\right) \\ \text{C. } 2P\left(Z \geq \left| z_{(\text{obs})} \right|\right) \end{cases}$

Statistical Inference About σ_1^2 / σ_2^2

Confidence interval for the ratio of the variances of two normal distributions:

$$\frac{S_1^2}{S_2^2} F_{1-\alpha/2, n_2-1, n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\alpha/2, n_2-1, n_1-1} \quad \text{(Normality is required.)}$$

Equivalent version : $$\frac{S_1^2}{S_2^2\ F_{\alpha/2, n_1-1, n_2-1}} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\alpha/2, n_2-1, n_1-1}$$

Upper confidence limit: $$\frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\alpha, n_2-1, n_1-1}$$

Lower confidence limit: $$\frac{\sigma_1^2}{\sigma_2^2} \ge \frac{S_1^2}{S_2^2\ F_{\alpha, n_1-1, n_2-1}}$$

$F_{0.10, \nu_1, \nu_2}$

Degrees of freedom for the numerator (n_1); Degrees of freedom for the denominator (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.86	1.84	1.81	1.78	1.75	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54	1.50
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
30	2.88	2.49	2.28	2.14	2.03	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00

Note: $F_{0.90, \nu_1, \nu_2} = 1/F_{0.10, \nu_2, \nu_1}$.

Statistical test about $\sigma_1^2 - \sigma_2^2$: (Normality is required.)

$H_0 : \sigma_1^2 - \sigma_2^2 = 0$

H_a : A. $\sigma_1^2 - \sigma_2^2 > 0$

B. $\sigma_1^2 - \sigma_2^2 < 0$

C. $\sigma_1^2 - \sigma_2^2 \neq 0$

Test Statistic: $F = \dfrac{S_1^2}{S_2^2}$

$$H_0 \text{ is rejected if } \begin{cases} \text{A. } F > F_{\alpha, n_1-1, n_2-1} \\ \text{B. } F < F_{1-\alpha, n_1-1, n_2-1} \\ \text{C. } F < F_{1-\alpha/2, n_1-1, n_2-1} \text{ or } F > F_{\alpha/2, n_1-1, n_2-1} \end{cases}$$

H_0 is rejected if A. $F > F_{\alpha, n_1-1, n_2-1}$,

B. $F < \dfrac{1}{F_{\alpha, n_2-1, n_1-1}}$,

C. $F < \dfrac{1}{F_{\alpha/2, n_2-1, n_1-1}}$ or $F > F_{\alpha/2, n_1-1, n_2-1}$.

$\left(\text{This is an alternative way to make decision.} \quad \left(\because F_{1-\alpha, m, n} = \dfrac{1}{F_{\alpha, n, m}}\right)\right)$

$$p\text{-value} = \begin{cases} \text{A. } P\left(F \geq F_{(\text{obs})}\right) \\ \text{B. } P\left(F \leq F_{(\text{obs})}\right) \\ \text{C. } 2 \min\left\{P\left(F \leq F_{(\text{obs})}\right), P\left(F \geq F_{(\text{obs})}\right)\right\} \end{cases}$$

Example (Continue) An article in the *Journal Hazardous Waste and Hazardous Materials* (Vol. 6, 1989) reported the results of an analysis of the weight of calcium in standard cement and cement doped with lead. Reduced levels of calcium would indicate that the hydration mechanism in the cement is blocked and would allow water to attack various locations in the cement structure. Ten samples of standard cement had an average weight percent calcium of $\bar{x}_1 = 90.0$, with a sample standard deviation of s_1=5.0, and 15 samples of the lead-doped cement had an average weight percent calcium of $\bar{x}_2$=87.0, with a sample standard deviation of s_2=4.0. It is assumed that weight percent calcium is normally distributed. Using $\alpha = 0.1$, check if there is any significant difference between two population variances.

Some Examples

Example A random sample of 400 nightlights was tested, and 40 lights were found defective. Construct a 90% confidence interval of the true fraction of defective nightlights.

Example In a discussion of SAT scores, someone comments: "Because only a minority of high school students take the test, the scores overestimate the ability of typical high school seniors. The mean SAT mathematics score is about 519, but I think that if all seniors took the test, the mean score would be no more than 450." A test was given to a random sample of 500 seniors from California. These students had a mean score of 461. Is this good evidence against the claim that the mean for all California seniors is no more than 450? Assuming that the population standard deviation is 100. Use $\alpha=0.01$.

Example Muzzle velocities of eight shells tested with a new gunpowder yield a sample mean of $2,959$ feet per second and a standard deviation of 39.4. The manufacturer claims that the new gunpowder produces an average velocity of no less than 3,000 feet per second. Does the sample provide enough evidence to contradict the manufacturer's claim? Use $\alpha = 0.05$.

Example A machine in a certain factory must be repaired if it produces more than 10% defectives among the large lot of items it produces in a day. A random sample of 100 items from the day's production contains 15 defectives, and the foreman says that the machine must be repaired. Does the sample evidence support his decision at the 0.01 significance level?

Example A hospital administrator suspects that the delinquency rate in the payment of hospital bills has increased over the past year. Hospital records show that the bills of 48 of 1284 persons admitted in the month of April have been delinquent for more than 90 days. This number compares with 34 of 1002 persons admitted during the same month one year ago. Do these data provide sufficient evidence to indicate an increase in the rate of delinquency in payments exceeding 90 days? Test using α=0.10.

Example A company employing a new sales-plus-commission compensation plan for its sales personnel wants to compare the annual salary expectations of its female and male personnel under the new plan. Random samples of 25 female and 20 male sales representatives were asked to forecast their annual incomes under the new plan. Sample means and standard deviations were

$$\bar{x}_1 = \$31083, \quad \bar{x}_2 = \$29745, \quad s_1 = \$2312, \quad s_2 = \$2569.$$

Do the data provide sufficient evidence to indicate a difference in mean expected annual income between female and male sales representatives? Test using α=0.05. It is assumed that the population distributions are normal and the population variances are the same.

Example A company employing a new sales-plus-commission compensation plan for its sales personnel wants to compare the annual salary expectations of its female and male personnel under the new plan. Random samples of 25 female and 20 male sales representatives were asked to forecast their annual incomes under the new plan. Sample means and standard deviations were

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$$\bar{x}_1 = \$31083, \ \ \bar{x}_2 = \$29745, \ \ s_1 = \$2312, \ \ s_2 = \$2569.$$

Do the data provide sufficient evidence to conclude that the two population variances are different? Test using α=0.05. It is assumed that the population distributions are normal.

Example A pharmaceutical company uses a machine to pour cold medicine into bottles in such a way that the standard deviation of the weights is 0.15 oz. A new machine is tested on 71 bottles, and the standard deviation for this sample is 0.12 oz. Is there enough evidence to conclude that the new machine can fill bottles with lower standard deviation? Use 0.05 as the significance level. It is assumed that the population distributions are normal.

Example Approximately 1 in 10 consumers favor cola brand A. After a promotional campaign in a sales region, 200 cola drinkers were randomly selected from consumers in the market area and were interviewed to determine the effectiveness of the campaign. The result of the survey showed that a total of 26 people expressed a preference for cola brand A. Do these data present sufficient evidence to indicate an increase in the acceptance of brand A in the region? Use α=0.05.

Example Alcohol abuse has been described by college presidents as the number one problem on campus, and it is an important cause of death in young adults. How common is it? A survey of 17,096 students in U.S. four-year colleges collected information on drinking behavior and alcohol-related problems. The researchers defined "frequent binge drinking" as having five or more drinks in a row three or more times in the past two weeks. According to this definition, 3314 students were classified as frequent binge drinkers. Find a 95% confidence interval for the proportion of frequent binge drinkers.

Type I Error, Type II Error and Power of Statistical Test

Example It is known that there are three balls in a bag. The balls may have red or green color.

H_0: There are more red balls than green balls in the bag. (2 or 3 red ball)

H_a: There are more green balls than red balls in the bag. (0 or 1 red ball)

One ball is drawn from the bag at random. Reject H_0 if the ball is green.

a. Find the level of significance of the test.

Example It is known that there are three balls in a bag. The balls may have red or green color.

H_0: There are more red balls than green balls in the bag. (2 or 3 red ball)

H_a: There are more green balls than red balls in the bag. (0 or 1 red ball)

One ball is drawn from the bag at random. Reject H_0 if the ball is green.

b Find the power of the test.

Example A certain machine manufactures parts. The machine is considered to be operating properly if 5% or less of the manufactured parts are defective. If more than 5% of the parts are defective, the machine needs remedial attention. A random sample of ten parts is selected to conduct the test.

$$H_0 : p \le 0.05 \quad \text{vs.} \quad H_a : p > 0.05$$

Let X be the number of defective parts in the sample. Reject H_0 when $X > 2$.

a. Find the level of significance of the test.

Example A certain machine manufactures parts. The machine is considered to be operating properly if 5% or less of the manufactured parts are defective. If more than 5% of the parts are defective, the machine needs remedial attention. A random sample of ten parts is selected to conduct the test.

$$H_0 : p \le 0.05 \quad \text{vs.} \quad H_a : p > 0.05$$

Let X be the number of defective parts in the sample. Reject H_0 when $X > 2$.

b Find the power of the test when $p = 0.1$ and $p = 0.2$.

Example The mean contents of coffee cans filled on a particular production line are being studied. Standards specify that the mean contents must be 16 oz, and from the past experience it is known that the standard deviation of the can contents is 0.1 oz. The hypotheses are:

$H_0: \mu = 16$ vs. $H_a: \mu \neq 16.$

A random sample of nine is used for the test using significance level $\alpha = 0.05$. It is assumed that the population distribution is normal.

General case:

A statistical test $H_0: \mu = \mu_0$ vs. $H_a: \mu \neq \mu_0$ is conducted. A random sample of size n is used for the test using significance level α. It is assumed that the population distribution is normal and that the population variance σ^2 is known.

The test statistic is $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$. H_0 is rejected if $|z_{(\text{obs})}| > z_{\alpha/2}$.

Find the probability of type II error and the power of the test if the true population mean is $\mu_1 = \mu + \delta$.

$$\beta(\mu_1) = P\left(-z_{\alpha/2} - \delta\frac{\sqrt{n}}{\sigma} \le Z \le z_{\alpha/2} - \delta\frac{\sqrt{n}}{\sigma}\right)$$

It can be seen that $\beta(\mu_1)$ is a functionof n, δ, and α for known σ.

Operating Characteristic Curves (O-C Curves)

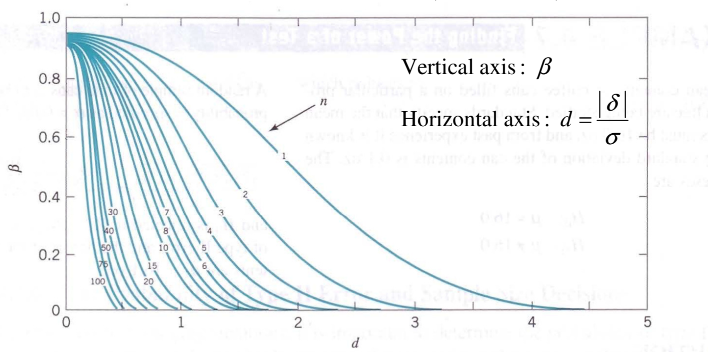

FIGURE 4.7 Operating-characteristic curves for the two-sided normal test with $\alpha = 0.05$. (Reproduced with permission from C. L. Ferris, F. E. Grubbs, and C. L. Weaver, "Operating Characteristic Curves for the Common Statistical Tests of Significance," *Annals of Mathematical Statistics*, June 1946.)

Comparing Multiple Population Means - ANOVA

Treatment	Observations				Total	Average
1	y_{11}	y_{12}	$\cdots$	y_{1n_1}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n_2}	$y_{2.}$	$\bar{y}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an_a}	$y_{a.}$	$\bar{y}_{a.}$
					$y_{..}$	$\bar{y}_{..}$

$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$

H_a : At least two treatment means differ.

$$SS_{\text{Total}} = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{..})^2$$

$$SS_{\text{Error}} = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i.})^2 = \sum_{j=1}^{n_1}(y_{1j} - \bar{y}_{i.})^2 + \sum_{j=1}^{n_2}(y_{2j} - \bar{y}_{i.})^2 + \cdots + \sum_{j=1}^{n_a}(y_{aj} - \bar{y}_{i.})^2$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} n_i(\bar{y}_{i.} - \bar{y}_{..})^2$$

$$SS_{\text{Total}} = SS_{\text{Error}} + SS_{\text{Treatments}}$$

$$\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i.})^2 + \sum_{i=1}^{a} n_i(\bar{y}_{i.} - \bar{y}_{..})^2$$

Source	d.f.	SS	MS	F
Treatments	$a-1$	$\sum_{i=1}^{a} n_i(\bar{y}_{i.} - \bar{y}_{..})^2$	$MS_{\text{Treatments}}$	
Error	$\sum_{i=1}^{a}(n_i - 1)$	$\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i.})^2$	MS_{Error}	
Total	$N-1$	$\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{..})^2$		

$N = \sum_{i=1}^{a} n_i$

Test statistic: $F = \dfrac{MS_{\text{Treatments}}}{MS_{\text{Error}}} = \dfrac{SS_{\text{Treatments}} \big/ (a-1)}{SS_{\text{Error}} \Big/ \left(\sum_{i=1}^{a}(n_i - 1)\right)}$

H_0 is rejected if $F_{(\text{obs})} > F_{\alpha,\, a-1,\, \sum_{i=1}^{a}(n_i-1)}$.

Treatment	Observations				Total	Average
1	y_{11}	y_{12}	$\cdots$	y_{1n_1}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n_2}	$y_{2.}$	$\bar{y}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an_a}	$y_{a.}$	$\bar{y}_{a.}$
					$y_{..}$	$\bar{y}_{..}$

Special case: $n_1 = n_2 = \cdots = n_a = n$

$$SS_{\text{Total}} = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2$$

$$SS_{\text{Error}} = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2 = \sum_{j=1}^{n}(y_{1j} - \bar{y}_{i.})^2 + \sum_{j=1}^{n}(y_{2j} - \bar{y}_{i.})^2 + \cdots + \sum_{j=1}^{n}(y_{aj} - \bar{y}_{i.})^2$$

$$SS_{\text{Treatments}} = n\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2$$

$$MS_{\text{Error}} = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2 \Big/ \left(a(n-1)\right)$$

Treatment	Observations				Total	Average
1	y_{11}	y_{12}	$\cdots$	y_{1n_1}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n_2}	$y_{2.}$	$\bar{y}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an_a}	$y_{a.}$	$\bar{y}_{a.}$
					$y_{..}$	$\bar{y}_{..}$

Special case: $n_1 = n_2 = \cdots = n_a = n$

Shortcut fomula:

$$SS_{\text{Total}} = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{an} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$SS_{\text{Treatments}} = n\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2 = \frac{\sum_{i=1}^{a} y_{i.}^2}{n} - \frac{y_{..}^2}{an} = \frac{\sum_{i=1}^{a} y_{i.}^2}{n} - \frac{y_{..}^2}{N}$$

$$SS_{\text{Error}} = SS_{\text{Total}} - SS_{\text{Treatments}}$$

$$SS_{\text{Total}} = SS_{\text{Error}} + SS_{\text{Treatments}}$$

$$\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2 + n\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2$$

Source	d.f.	SS	MS	F
Treatments	$a-1$	$n\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2$	$MS_{\text{Treatments}}$	
Error	$a(n-1)$	$\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2$	MS_{Error}	
Total	$N-1$	$\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2$		

$$\text{Test statistic}: \ F = \frac{MS_{\text{Treatments}}}{MS_{\text{Error}}} = \frac{SS_{\text{Treatments}}/(a-1)}{SS_{\text{Error}}/\left(a(n-1)\right)}$$

H_0 is rejected if $F_{(\text{obs})} > F_{\alpha,\,a-1,\,a(n-1)}$.

Example A manufacturer of paper used for making grocery bags is interested in improving the tensile strength of the product. Product engineering thinks that tensile strength is a function of the hardwood concentration in the pulp and that the range of hardwood concentrations of practical interest is between 5% and 20%. A team of engineers responsible for the study decides to investigate four levels of hardwood concentrations: 5%,, 10%, 15%, and 20%. They decide to make up six test specimens at each concentration level, using a pilot plant. All 24 specimens are tested on a laboratory tensile tester, in a random order. The data are shown in the table.

Hardwood Concentration (%)	Observations					
	1	2	3	4	5	6
5	7	8	15	11	9	10
10	12	17	13	18	19	15
15	14	18	19	17	16	18
20	19	25	22	23	18	20

Use the analysis of variance to test the hypothesis that different hardwood concentrations do not affect the mean tensile strength of the paper.

Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6
5	7	8	15	11	9	10
10	12	17	13	18	19	15
15	14	18	19	17	16	18
20	19	25	22	23	18	20

Source	d.f.	SS	MS	F
Treatments	3	382.79	127.5967	19.6046
Error	20	130.17	6.5085	
Total	23	512.96		

Treatment	Observations				Total	Average
1	y_{11}	y_{12}	$\cdots$	y_{1n_1}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n_2}	$y_{2.}$	$\bar{y}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an_a}	$y_{a.}$	$\bar{y}_{a.}$
					$y_{..}$	$\bar{y}_{..}$

	Estimated	Residual
$Y_{1j} = \mu_1 + \varepsilon_{1j} \ \ (j = 1, 2, \cdots, n_1)$	$\hat{\mu}_1 = \bar{Y}_{1.}$	$e_{1j} = Y_{1j} - \bar{Y}_{1.} \ \ (j = 1, 2, \cdots, n_1)$
$Y_{2j} = \mu_2 + \varepsilon_{2j} \ \ (j = 1, 2, \cdots, n_2)$	$\hat{\mu}_2 = \bar{Y}_{2.}$	$e_{2j} = Y_{2j} - \bar{Y}_{2.} \ \ (j = 1, 2, \cdots, n_2)$
$\vdots$	$\vdots$	$\vdots$
$Y_{aj} = \mu_a + \varepsilon_{aj} \ \ (j = 1, 2, \cdots, n_a)$	$\hat{\mu}_a = \bar{Y}_{a.}$	$e_{aj} = Y_{aj} - \bar{Y}_{a.} \ \ (j = 1, 2, \cdots, n_a)$

Residual Plot